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MINIMAL PAIRS OF POLYTOPES AND THEIR NUMBER OF VERTICES

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ABSTRACT: We define what is called Blaschke difference for polytopes as an inverse operation to Blaschke addition. Using this operation we give a new algorithm to reduce and find a minimal pair of polytopes from the given class of the Rådström-Hörmander lattice containing a pair of polytopes in $\mathbb{R}^2$. This method gives a better algorithmic insight and easy to handle than the one given by Handschug (1989). We also prove that a pair of polytopes in the plane is minimal if and only if the sum of the number of their vertices is minimal in the class. However, it is shown in the paper that, this last statement does not hold true in general for higher dimensional spaces.

Key words/phrases: Pairs of compact convex sets, Blaschke addition, Minkowski sum, mnimality, quasidiffenertial calculus

INTRODUCTION

Vector addition of convex compact sets (called Minkowski addition) has proved to be useful in different investigations. However there are many useful properties which fail to hold with this addition. For example, if a set A is not singleton, $A-A$ is different from {0} on any dimension; moreover, decomposability of polytopes does not hold true when we consider vector spaces of dimensions greater than 2. To this effect we need the composition of convex compact sets, known as **Blaschke addition**, which overcomes at least the above mentioned limitations. It was first proposed by Blaschke (1916) and used by various authors for different purposes (*e.g.*, Firey and Grünbaum, 1964; Schneider, 1993; Bonetti and Vitale, 2000). This addition can also be used to find a minimal representative from a given equivalent class of the Rådström-Hörmander lattice, which will be our main focus on this paper.

Let X be a real topological vector space and $\mathcal{K}(X)$ denote the set of nonempty convex compact subsets of X endowed with the usual Minkowski addition and scalar multiplication. The set $\mathcal{K}(X)$ is a commutative semigroup with cancellation property (Urbański, 1976). Denote the cartesian product $\mathcal{K}(X) \times \mathcal{K}(X)$ by $\mathcal{K}^2(X)$.

Like in Pallaschke and Urbański (1994) we say that two pairs (A,B) and (C,D) from $\mathcal{K}^2(X)$ are **equivalent**, written $(A,B)\sim(C,D)$ if, and only if $A+D=B+C$ with the Minkowski addition. Let us recall that the quotient space $\mathcal{K}^2(X)/\sim$ is called the Rådström-Hörmander lattice. We denote by $[A,B]$ the equivalence class determined by (A,B). We can define the ordering relation $\leq$ on $\mathcal{K}^2(X)$ by $(A,B)\leq(C,D)$ iff $(A,B)\sim(C,D)$ and $A\subseteq C$, $B\subseteq D$.

The pair $(A,B) \in \mathcal{K}^2(X)$ is said to be **minimal** if it is minimal in the class $[A,B]$ with respect to the above ordering relation. Minimal pairs of compact convex sets have been investigated since the 1980's mainly by Pallaschke and Urbanski (see for instance Pallaschke *et al.*, 1991; Scholtes, 1992; Grzybowski, 1994; Pallaschke and Urbański, 1996); different authors also proved some minimality criteria.

BLASCHKE ADDITION

The key to this addition operation is the existence theorem of Minkowski (1903) and Schneider (1993). Before we state the theorem, we first sketch the notion of **surface area measure** φ for a convex compact set of $K\subset\mathbb{R}^n$ (called the surface area measure of order n–1). For each Borel set $\mathcal{U}\subseteq S^{n-1}$ where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, let $bdK(\mathcal{U})$ be the set of points in boundary of K having outward normal vectors in $\mathcal{U}$. Then (see Schneider, 1993) the surface area measure φ is defined on the Borel σ-field of S^{n-1} via $\varphi(\mathcal{U})=\lambda_{n-1}(bdK(\mathcal{U}))$, where λ_{n-1} is (n–1)-dimensional Hausdorff measure. The surface area measure of K is also denoted by $S_{n-1}(K,.)$ and is a finite measure on S^{n-1}.

If φ is the area measure of a convex compact set then (Schneider, 1993:281)

$$\int_{S^{n-1}} u\,d\varphi = 0$$

The general Minkowski theorem is then the converse of the above equation, which is stated below.

Theorem 1 *Suppose that φ is a bounded measure on the Borel subset of S^{n-1} that is not concentrated on a great sphere and that satisfies*

$$\int_{S^{n-1}} u\,d\varphi = 0$$

Then there is a unique (up to translation) convex body for which φ is the surface area measure.

With the help of this Minkowski theorem it is possible now to define what is called Blaschke addition of two compact convex sets. Let $L,K \subset IR^n$ be compact convex sets. Then there exists a convex compact set $M \subset IR^n$, such that

$$S_{n-1}(L,\cdot) + S_{n-1}(K,\cdot) = S_{n-1}(M,\cdot).$$

This set M is called the **Blaschke sum** of L and K, denoted by $L\#K$, and is determined only up to translation.

We will illustrate this addition for the case of polytopes in concrete ways (see Firey and Grünbaum, 1964). Let P be a k-polytope in IR^n and let IR^k be the k-dimensional subspace parallel to the affine hull of P (aff P). Denote by $f_{k-1}(P) \geq k+1$ the number of facets of P, and with each facet $F_i (1 \leq i \leq f_{k-1}(P))$ associate its outward unit normal vector u_i. Then with every k-polytope P, $k \geq 2$ we can associate the system of vectors

$$V(P) = \{u_i V_{k-1}(F_i) \mid 1 \leq i \leq f_{k-1}(P)\}, \tag{1}$$

where $V_{k-1}(F_i)$ denotes the volume of facet F_i (which is (k–1) dimensional set). If polytopes P_1 and P_2 are translates of each other, then clearly $V(P_1) = V(P_2)$. Therefore, $V(P)$ can be considered as being associated with the translation class of polytopes containing P.

If Q is a (k–1)-dimensional polytope in IR^k, we define a set of vectors

$$V(Q) = \{u_0 V_{k-1}(Q), -u_0 V_{k-1}(Q)\}, \tag{2}$$

where u_0 is a unit vector normal to the hyperplane (in IR^k) containing Q to be associated with Q.

Definition 2 *(See Grünbaum, 1967) A system $W = \{v_i | 1 \leq i \leq m\}$ of nonzero vectors in IR^k is called equilibrated if*

$$\sum_{i=1}^{m} v_i = 0 \tag{3}$$

and no two of the vectors of $\mathcal{W}$ are parallel. $\mathcal{W}$ is called fully equilibrated in IR^k provided it is equilibrated and spans IR^k.

With this terminology, we now restate the Minkowski existence theorem for polytopes.

Theorem 3 (Minkowski's Theorem)

(i) If P is a polytope in IR^n, then $V(P)$ is equilibrated. If P is a k-polytope, then $V(P)$ is fully equilibrated in the subspace IR^k parallel to the affine hull of P.

(ii) If $\mathcal{W}$ is a fully equilibrated system of nonzero vectors in $IR^k (k \geq 2)$, then there exists a polytope P, unique up to translation, such that $\mathcal{W}=V(P)$.

For the proof of this theorem the reader is referred to (Grünbaum, 1967:339–340). We now illustrate the Blaschke sum of two polytopes and also define a Blaschke difference, a composition which can be viewed as an inverse operation to the Blaschke addition.

Let $A,B \in \mathcal{K}^2(IR^n), n \geq 2$ be polytopes of dimensions k_1 and k_2 respectively, $U(A)$ denotes a set of unit normal vectors of a polytope A at its facets, and let $|X|$ denotes the cardinality of the set X.

Let

$$V(A) = \{u_i V_{k_1-1}(F(A,u_i)) \mid 1 \leq i \leq f_{k_1-1}(A)\} \text{ and}$$

$$V(B)=\{v_i V_{k_2-1}(F(B,v_i))\,|\,1\le i\le f_{k_2-1}(B)\}$$

Put

$\mathcal{W}=\{e_j d_j\,|\,1\le j\le f_{k_1-1}(A)+f_{k_2-1}(B)-|U(A)\cap U(B)|\}$,

where,

$$e_j=\begin{cases} w=u_i=v_i \text{ for some } i, & \text{if } w\in\mathcal{U}(A)\cap\mathcal{U}(B)\\ u_j, & \text{if } u_j\in\mathcal{U}(A)\setminus\mathcal{U}(B)\\ v_j, & \text{if } v_j\in\mathcal{U}(B)\setminus\mathcal{U}(A)\end{cases}$$

and

$$d_j=\begin{cases} V_{k_1-1}(F(A,u))+V_{k_2-1}(F(B,u)), & \text{if } u\in\mathcal{U}(A)\cap\mathcal{U}(B)\\ V_{k_1-1}(F(A,u)), & \text{if } u\in\mathcal{U}(A)\setminus\mathcal{U}(B)\\ V_{k_2-1}(F(B,u)), & \text{if } u\in\mathcal{U}(B)\setminus\mathcal{U}(A)\end{cases}$$

Then since both $V(A)$ and $V(B)$ are equilibrated, V is also equilibrated. Moreover, the affine hull of V is of dimension $k\ge\max\{k_1, k_2\}$. Thus $\mathcal{W}$ is fully equilibrated in some space IR^k. Therefore, by Minkowski theorem there is a polytope $P\subset IR^k$ such that $\mathcal{W}=V(P)$ and this P is unique up to translation. *i.e.*,

$$P=A\#B.$$

If $D=A\#B$, then clearly $U(A)\subseteq U(D)$ and $V_{k_1-1}(F(A,w))\le V_{k-1}(F(D,w))\quad\forall w\in U(A)$. Thus we can define a subtraction operation $\boxminus$ as follows:

Suppose $D=A\#B$ is given. Then we have

$$V(B)=\{e\alpha\,|\,e\in S^{n-1},\alpha\in IR\},$$

where

$$e=\begin{cases} u\in\mathcal{U}(D)\setminus\mathcal{U}(A), & \text{if } \nexists w\in\mathcal{U}(A) \text{ with } w=u\\ w, & \text{if } \exists u\in\mathcal{U}(A),\ \exists v\in\mathcal{U}(D) \text{ with } w=u=v \text{ and}\\ & V_{k_1-1}(F(A,w))\le V_{k-1}(F(D,w))\end{cases}$$

and

$$\alpha=\begin{cases} V_{k_1-1}(F(D,u)), & \text{if } u\in\mathcal{U}(D)\setminus\mathcal{U}(A)\\ V_{k_1-1}(F(D,u))-V_{k_1-1}(F(A,u)), & \text{if } u\in\mathcal{U}(A)\cap\mathcal{U}(D)\end{cases}$$

Hence the polytope B, which is determined (up to translation) by the set V(B) is the **Blaschke difference** of D and A. We shall denote this difference by

$$B=D\boxminus A$$

It is now an elementary exercise to prove the following assertions.

Proposition 4 *Let A,B and D be polytopes in* IR^n.

1. $A\boxminus A=\{0\}$ *for any polytope* $A\subset IR^n$.

2. $U(A\#B)=\{w\in U(A)|\nexists\ u\in U(B) \text{ with } u=w\}$
$\cup\{u\in U(B)\,|\nexists\ w\in U(A) \text{ with } w=u\}$
$\cup\{v\,|\,\exists w\in U(A),\exists u\in U(B) \text{ with } w=u=v\}$

3. *A is a Blaschke-summand*[1] *of D if and only if there exists a polytope B, which is the Blaschke-difference of D and A, i.e.* $B=D\boxminus A$. *In this case we have*

$$\mathcal{U}(B)=\{v\in\mathcal{U}(D)\,|\,\nexists u\in\mathcal{U}(B) \text{ with } u=w\}$$
$$\cup\{u\,|\,\exists w\in\mathcal{U}(A),\exists v\in\mathcal{U}(D) \text{ with } w=u=v \text{ and}$$
$$V_{k_1-1}(F(A,w))\le V_{k-1}(F(D,w))\}$$

One of the powerful results we get when we use Blaschke addition is the decomposition of polytopes with simplices (see Theorem 1 in Firey and Grünbaum, 1964), which has no analogue in Minkowski sum for dimensions greater than 2.

Moreover, since the surface area measure of order (n–1) and the area measure of order 1 coincide in IR^2, one can utilize Blaschke addition to develop a reduction algorithm to find a minimal pair of compact convex sets in the plane. Handschug (1989) developed the first method of this kind for polytopes in the plane, and later on Demyanov and Abankin (1997) produced a similar algorithm for the so called piecewise smooth sets in the plane. In both cases, however, the Blaschke sum is not mentioned though it is applied in different forms. The fact that these two area measures coincide in IR^2 assures the uniqueness up to translation of a minimal pair in the plane. This coincidence of the two measures in the plane was employed as well by Bauer (1996) in proving a strong criteria for minimality of pairs of compact convex sets in IR^2. Since the surface area measure S_{n-1} of order n–1 is not distributive over Minkowski addition in general for n>2, the above mentioned nice applications of Blaschke sum cannot be used to generalize the results obtained

[1] A compact convex set K is said to be a Blaschke-summand of M, if there exists a compact convex set L such that $M=K\#L$.

for pairs of compact convex sets in IR^2 in higher dimensions.

REVISED HANDSCHUG'S ALGORITHM

In this section we give a variant of Handschug's Algorithm using the above defined Blaschke sum and difference for polytopes in the plane. But first we need to prove the following Lemma.

Lemma 5 *The pair (A_1, B_1) is equivalent to (A_2, B_2) if and only if there exist convex compact sets C_1 and C_2 such that*

$$A_1+C_1=A_2+C_2 \text{ and } B_1+C_1=B_2+C_2 \qquad (4)$$

Proof: ▸ Suppose (A_1, B_1) and (A_2, B_2) are equivalent. Then by definition $A_1+B_2=B_1+A_2$. Then put $C_1=B_2$ and $C_2=B_1$ or $C_1=A_2$ and $C_2=A_1$. This proves one side of the implication.

For the converse, adding up the two equations in (4) we get

$$A_1 + B_2 + C_1 + C_2 = B_1 + A_2 + C_1 + C_2$$

Then by the cancellation law we have the expected result. ◂

Using this Lemma, it is possible to reduce the pair of compact convex sets (A_1, B_1) into an equivalent pair of compact convex sets (A_2, B_2), if we can find a compact convex set C_1 as small as possible and a compact convex set C_2 as large as possible. Hence, the problem of finding a minimal representative from the class of convex compact sets can be transformed into the problem of summands and decompositions of compact convex sets.

For a polytope A in IR^n we denote by $\mathcal{F}_1(A)$ the set of all edges (which are, one dimensional faces) of A. Let A,B be polytopes in IR^n. We say that $F\in\mathcal{F}_1(A)$ and $G\in\mathcal{F}_1(B)$ are **equiparallel** (as in Bauer (1996)) if F and G are parallel and if there is $u\in S^{n-1}$ with $F=F(A,u)$ and $G=F(B,u)$. The following theorem gives us the necessary and sufficient condition for a pair of polytopes in the plane to be minimal. The theorem is proved using surface area measure even in a more general setting for any convex compact sets in IR^2.

Theorem 6 *Let A,B be polytopes in IR^2, where A and B are not both straight lines. The pair $(A,B)\in\mathcal{K}^2_c(IR^2)$ is minimal iff A and B have at most one pair of equiparallel edges.*

For the proof of this theorem we refer to Corollary 3.6 in Bauer (1996). This theorem is a fundamental characterization of minimal pairs of polytopes in the plane.

Since Blaschke addition coincides (up to translation) with Minkowski addition on the plane, we can rewrite equation (4) equivalently (up to translation) as:

$$A_1\#C_1 = A_2\#C_2 \text{ and } B_1\#C_1 = B_2\#C_2 \qquad (5)$$

Or using Blaschke difference

$$A_2 = (A_1\#C_1) \boxminus C_2 \text{ and } B_2 = (B_1\#C_1) \boxminus C_2 \qquad (6)$$

But this last equation is equivalent to the statements: $\forall w\in U(C_2)\exists u\in U(A_1\#C_1)$ with w=u and $V_{k_2-1}(F(C_2,w)) \le V_{k-1}(F(A_1\#C_1,w))$ and $\forall w\in U(C_2)\exists v\in U(B_1\#C_1)$ with w=v and $V_{k_2-1}(F(C_2,w)) \le V_{k-1}(F(B_1\#C_1,w))$.

That means, the elements of the set $U(C_2)$ are from the intersection of the sets $U(A_1\#C_1)$ and $U(B_1\#C_1)$. Our aim, here, is to find a polytope C_2 as large as possible and a polytope C_1 as small as possible such that C_1 and C_2 satisfy relation (5). To this end, define a set of vectors,

$$V_0 = \{\alpha w \mid \exists u\in U(A_1), \exists v\in U(B_1) \text{ with } w=v=u \text{ and } \alpha=\min\{V_{n-1}(F(A_1,u)), V_{n-1}(F(B_1,v))\}\} \qquad (7)$$

and the vector

$$v_o = \sum_{\alpha w\in V_o} w \text{ if } V_o \neq \varnothing \text{ and } v_o = 0, \text{ otherwise.} \qquad (8)$$

Then put

$$V(C_1)=\{\beta e_o, -\beta e_o\} \text{ and } V(C_2)=V_o\cup\{-\beta e_o\} \text{ if } v_o\neq 0 \qquad (9)$$

where

$\beta = \|v_0\| \sum_{\alpha w\in V_0} \alpha$ and $e_0 = \dfrac{v_0}{\|v_0\|}$, with $\|\bullet\|$ denoting the Euclidean norm, or

$$V(C_1)=\{0\} \text{ and } V(C_2)=V_o \text{ if } v_o=0. \qquad (10)$$

To summarize the above discussion, if a pair of polytopes (A,B) from the class [A,B] is given, then we can reduce this pair to an equivalent and minimal pair of polytopes (A_0, B_0) (which is unique up to translation) using the following algorithm.

Step 1. Describe the polytopes A and B as sets of vectors

$V(A)=\{uV_{k_1-1}(F(A,u)) | u\in \mathcal{U}(A)\}$

$V(B)=\{vV_{k_2-1}(F(B,v)) | v\in U(B)\}$

Step 2. Collect the elements from both sets having the same direction with the minimum magnitude, say

$$V_0=\{aw \mid \exists u\in U(A), \exists v\in U(B) \text{ with } w=v=u \text{ and } \alpha=\min\{V_{n-1}(F(A,u)), V_{n-1}(F(B,v))\}\} \quad (11)$$

Step 3. Check for the sum $w_o=\sum_{w\in V_o} w$ of the vectors in V_O.

- If the sum is 0, put $V(C_2)=V_0$ and $V(C_1)=\{0\} \leftarrow$ (represents a point set)
- Else put $V(C_2)=V_0\cup\{-w_0\}$ and $V(C_1)=\{w_0,-w_0\} \leftarrow$ (represents a line)

Step 4. Determine the reduced polytopes by computing $A_o=(A\#C_1)\boxminus C_2$ and $B_o=(B\#C_1)\boxminus C_2$

Then clearly $V(C_1)$ and $V(C_2)$ as defined in equations (9) or (10), satisfy Minkowski's Theorem. Hence we can reconstruct the desired sets C_1 and C_2. Once $V(C_1)$ and $V(C_2)$ are calculated one can determine A_2 and B_2 from equation (6).

If the cardinality of V_0 is less than or equal to 1 then the pair (A_1, B_1) is minimal by Theorem 6. Otherwise, A_2 will contain at most one unit normal vector which is parallel to that of B_2 since all vectors of A_1 and B_1 having parallel unit normal vectors are collected in the set V_0. Hence A_2 and B_2 after reduction will have at most one pair of edges which are equiparallel. The above algorithm, therefore, yields a minimal pair (A_2, B_2).

MINIMALITY OF VERTICES OF PAIRS OF POLYTOPES IN IR^2

A polytope can be described as the convex hull of its vertices. Therefore, identifying minimality of pairs of polytopes with the number of their vertices will be more useful in application. The next theorem states that a pair of polytopes is minimal in its equivalence class if and only if the sum of the number of vertices of the polytopes is minimal in the class.

Proposition 7 *Let A,B be polytopes in IR^2 and let [A,B] denote the equivalence class determined by (A,B) in Rådström-Hörmander Lattice. If (A',B')∈[A,B] is minimal, then there is no pair (C,D)~(A',B') such that $|\mathcal{E}(C)|+|\mathcal{E}(D)|<|\mathcal{E}(A')|+|\mathcal{E}(B')|$, where $\mathcal{E}(A')$ denotes the set of extremal points of A' and |X| denotes the cardinal number of the set X.*

Proof: ▸ Let (C,D)∈[A,B] and assume (C,D) is not minimal. Then applying the above algorithm we can find a minimal pair (C_1, D_1) which is equivalent to (C,D) and unique up to translation.

Hence

$|V(C)|+|V(D)| \not< |V(C_1)|+|V(D_1)|=|V(A')|+|V(B')|$

which is equivalent to the relation

$|\mathcal{E}(C)|+|\mathcal{E}(D)| \not< |\mathcal{E}(A')|+|\mathcal{E}(B')|$. ◂

Theorem 8 *Let $A,B\subset IR^2$ be polytopes. Then the pair (A,B) is minimal only if $|\mathcal{E}(A)|+|\mathcal{E}(B)|$ is minimal for all (A,B)∈[A,B], where $\mathcal{E}(A)$ denotes the set of all extremal points of A and |X| denotes the cardinal number of the set X.*

Proof : ▸ If the given pair (*A*,*B*) is minimal, then by virtue of the algorithm described in the previous section there is no pair $(A_1, B_1)\in[A,B]$ such that

$$|V(A)|+|V(B)|>|V(A_1)|+|V(B_1)|,$$

where $|V(A)|$ denotes the cardinal number of the set *V*(*A*). For otherwise we can farther reduce the pair (*A*,*B*) to (A_1, B_1), which is not possible as (*A*,*B*) was chosen to be minimal. That means the number of edges of *A* and *B* is always less than that of

A_1 and B_1 for all pairs $(A_1,B_1)\in[A,B]$. In IR^2 since the number of edges of a polytope is equal to the number of its vertices, the assertion follows. ◂

Bounded polyhedral convex sets are easier to describe, using their facial structure, than any other compact convex sets. In the plane, equivalent classes of pairs of compact convex sets have some interesting properties in this direction. The idea of the following theorem can be seen as a mere consequence of the translation equivalence of two minimal pairs in the plane, which was proved independently by Scholtes (1992) and Grzybowski (1994). However, for completeness we give here a different proof by using the above given algorithm.

Theorem 9 *If A and B are two polytopes in the plane and C and D are any compact convex sets in IR^2 such that*

(i) (A,B)~(C,D) and

(ii) (C,D) is minimal

Then C and D are also polytopes.

Proof: ▸ Let A and B be polytopes and assume without loss of generality that the pair (A,B) is not minimal. Then using the above given algorithm (Handschug's Algorithm) we can reduce the pair to get a minimal pair of polytopes, say (A_1,B_1), which is equivalent to (A,B).

But since $(A,B)\sim(C,D)$ and $(A,B)\sim(A_1,B_1)$ we have $(A_1,B_1)\sim(C,D)$.

From assumption *(ii)* we know that (C,D) is minimal and since minimal pairs in the plane are unique up to translations (Scholtes, 1992; Grzybowski, 1994), the assertion of the theorem follows. ◂

More strongly we have the following corollary.

Corollary 10 *If an equivalence class [A,B] of compact convex sets in the plane contains at least one pair of polyhedral sets, then any minimal pair is also polyhedral.*

Proof: ▸ Let $C,D\in K(IR^2)$ be polytopes and $(C,D)\in[A,B]$. Suppose (C,D) is not minimal. Since every equivalence class $[A,B]$ contains a minimal element, we can find some $(A_1,B_1)\in[A,B]$ which is minimal. Then we can apply Theorem 9 above to find the conclusion of the corollary. ◂

Therefore, if an element in Rådström-Hörmander lattice in IR^2 contains at least one pair of polytopes, then the minimal pairs definitely have minimal number of extremal points.

RELATIONSHIP BETWEEN MINIMALITY OF PAIRS OF POLYTOPES AND THEIR NUMBER OF VERTICES

For the case when the dimension of the underlying space is less than or equal to 2, we have proved in Theorem 8 that the statement conjectured by Pallaschke and Urbański[2] works perfectly. Moreover, the conjecture is proved to be true for reduced pairs in higher dimensions as well (see Theorem 12 below).

Lemma 11 *Let $A,B\subset IR^n$ be compact convex sets. For any $a\in\mathcal{E}(A)$ there exists $b\in\mathcal{E}(B)$ such that $a+b\in\mathcal{E}(A+B)$.*

Proof: ▸ The proof goes by induction on n (dimension of the space IR^n). The assertion is obviously true for n=1, because convex compact sets in IR are closed intervals, and end points of the interval of the sum is the sum of the end points of the component intervals. Let us assume that the lemma holds true for $n=1,2,\cdots,k-1$ and suppose $A,B\subset IR^k$, $a\in\mathcal{E}(A)$. Then there exists a facet of A containing a. Let this facet be determined by the normal vector $u\in IR^k$. Now consider the sets $F(A,u)$ and $F(B,u)$. These two sets are nonempty, compact convex sets contained in parallel hyperplanes and every extreme points of

[2]**Conjecture**(Pallaschke and Urbański): If a pair of polytopes is minimal then the sum of the number of their vertices is minimal.

$F(A,u)$ and $F(B,u)$ are extreme points of A and B, respectively. Moreover we have $F(A+B,u)=F(A,u)+F(B,u)$. For suitable translations $F(A,u)$ and $F(B,u)$ are contained in a $(k-1)$-dimensional subspace of IR^k. Hence by the assumption, for the extreme point a of $F(A,u)$ there exists an extreme point b of $F(B,u)$ such that $a+b\in \mathcal{E}(F(A+B,u))$. Hence the lemma. ◂

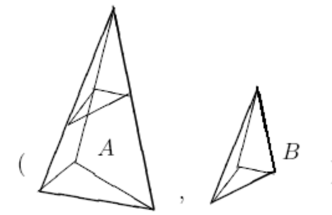
In Bauer (1996) it is defined that a pair of compact convex sets (A,B) is said to be **reduced** if and only if every element of [A,B] can be described as (A+K,B+K) for some convex compact set K. It is easy to verify that all reduced pairs are minimal. Moreover, reduced pairs are the up to translation unique minimal pairs in their class.

The extremal points of the sum $A+B$ are the only points having a unique representation as a sum of two points from A and B. Indeed if $a+b = a'+b'\in\mathcal{E}(A+B)$, then $a+b=\frac{1}{2}(a+a')+\frac{1}{2}(b+b')\in\mathcal{E}(A+B)$ which implies that $a=a'$ and $b=b'$. This discussion will lead us to the next theorem, in which we will show that if a pair of any compact convex sets is reduced, then the sum of their extremal points is minimal in the class.

Theorem 12 *Let* $A,B\subset IR^n$ *be compact convex sets. If the pair (A,B) is reduced then* $|\mathcal{E}(A)|+|\mathcal{E}(B)|\le |\mathcal{E}(C)|+|\mathcal{E}(D)|$ for all $(C,D)\sim(A,B)$.

Proof: ▸ Let (A,B) be reduced. Then $(C,D)\sim(A,B)$ implies that there exists a compact convex set $M\subset IR^n$ such that $C=A+M$ and $D=B+M$. Then by Lemma 11, we have that for any extremal point a of A, there always exist an extremal point, say m, of M such that $a+m$ is an extremal point of C. Since this representation is unique (by the remark prior to this theorem) we can conclude that $|\mathcal{E}(A)|\le|\mathcal{E}(C)|$. In a similar fashion we also have $|\mathcal{E}(B)|\le|\mathcal{E}(D)|$. Hence $|\mathcal{E}(A)|+|\mathcal{E}(B)| \le|\mathcal{E}(C)|+|\mathcal{E}(D)|$. ◂

However, in dimensions greater than or equal to 3, this nice relationship between the number of vertices and the minimality of a pair of polytopes does not hold true in general. The following simple counter example shows this fact (Fig. 1).



Can be reduced using cutting hyperplane method to a pair:

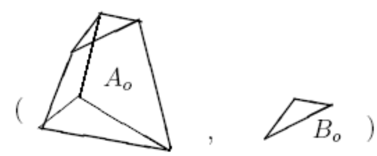


Fig. 1. Minimal pair of polytopes with more number of vertices.

The pair (A_0,B_0) in Fig. 1 which is equivalent to the pair (A,B) is non-reducible and it is minimal as well by Theorem 5.1 in Bauer (1996). However

$$|\mathcal{E}(A)|+|\mathcal{E}(B)|=4+4<6+3=|\mathcal{E}(A_0)|+|\mathcal{E}(B_0)|$$

As an application to Theorem 8, we consider the following problem. Let a function $f: IR^2\to IR$ be piecewise-linear and continuous. Then it is shown by Melzer (1986) that f can be written as a difference of two piecewise-linear convex functions as:

$$f(x) := \max_{u\in\underline{\partial} f(0)}\langle u,x\rangle - \max_{v\in\overline{\partial} f(0)}\langle v,x\rangle \qquad (12)$$

But this representation is not unique and also one needs a minimal representation of the difference in equation (12). It is clear that the sets $\underline{\partial} f(0)$ and $\overline{\partial} f(0)$ are polytopes in IR^2. Thus by Theorem 8 a pair (A,B) which is equivalent to $(\underline{\partial} f(0),\overline{\partial} f(0))$, is minimal, if and only if the sum of vertices, say n_A and n_B, of A and B, respectively, is minimal. Thus equation (12) is equivalent to:

$$f(x)=\max_{i\in\{1,\cdots,n_A\}}\langle a_i,x\rangle - \max_{j\in\{1,\cdots,n_B\}}\langle b_j,x\rangle,$$

where n_A+n_B is minimal among all such expressions.

Note that since the proofs of the above characterizations are based on Handschug's Algorithm, which in turn is based on the equivalence (up to translation) of Blaschke sum to that of Minkowski sum, it is applicable only for 2 dimensional real vector spaces or for reduced pairs in dimensions higher than 2. For non reduced pairs in higher dimensions this relation does not hold in general as indicated with the above counter example in 3-dimensional case.

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